

RANDOM NORMAL MATRICES AND WARD IDENTITIES

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ABSTRACT. Consider the random normal matrix ensemble associated with a potential on the plane which is sufficiently strong near infinity. It is known that, to a first approximation, the eigenvalues obey a certain equilibrium distribution, given by Frostman's solution to the minimum energy problem of weighted logarithmic potential theory. On a finer scale, one can consider fluctuations of eigenvalues about the equilibrium. In the present paper, we give the correction to the expectation of fluctuations, and we prove that the potential field of the corrected fluctuations converge on smooth test functions to a Gaussian free field with free boundary conditions on the droplet associated with the potential.

Given a suitable real "weight function" in the plane, it is well-known how to associate a corresponding (weighted) random normal matrix ensemble (in short: RNM-ensemble). Under reasonable conditions on the weight function, the eigenvalues of matrices picked randomly from the ensemble will condensate on a certain compact subset S of the complex plane, as the order of the matrices tends to infinity. The set S is known as the *droplet* corresponding to the ensemble. It is well-known that the droplet can be described using weighted logarithmic potential theory and, in its turn, the droplet determines the classical equilibrium distribution of the eigenvalues (Frostman's equilibrium measure).

In this paper we prove a formula for the expectation of fluctuations about the equilibrium distribution, for linear statistics of the eigenvalues of random normal matrices. We also prove the convergence of the potential fields corresponding to corrected fluctuations to a Gaussian free field on S with free boundary conditions.

Our approach uses *Ward identities*, that is, identities satisfied by the joint intensities of the point-process of eigenvalues, which follow from the reparametrization invariance of the partition function of the ensemble. Ward identities are well known in field theories. Analogous results in random Hermitian matrix theory are known due to Johansson [13], in the case of a polynomial weight.

General notation. By $D(a, r)$ we mean the open Euclidean disk with center a and radius r . By "dist" we mean the Euclidean distance in the plane. If A_n and B_n are expressions depending on a positive integer n , we write $A_n \lesssim B_n$ to indicate that $A_n \leq CB_n$ for all n large enough where C is independent of n . The notation $A_n \asymp B_n$ means that $A_n \lesssim B_n$ and $B_n \lesssim A_n$. When μ is a measure and f a μ -measurable function, we write $\mu(f) = \int f d\mu$. We write $\partial = \frac{1}{2}(\partial/\partial x - i\partial/\partial y)$ and $\bar{\partial} = \frac{1}{2}(\partial/\partial x + i\partial/\partial y)$ for the complex derivatives.

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1. RANDOM NORMAL MATRIX ENSEMBLES

1.1. The distribution of eigenvalues. Let $Q : \mathbb{C} \rightarrow \mathbb{R} \cup \{+\infty\}$ be a suitable lower semi-continuous function subject to the growth condition

$$(1.1) \quad \liminf_{z \rightarrow \infty} \frac{Q(z)}{\log |z|} > 1.$$

We refer to Q as the *weight function* or the *potential*.

Let \mathcal{N}_n be the set of all $n \times n$ normal matrices M , i.e., $MM^* = M^*M$. The *partition function* on \mathcal{N}_n associated with Q is the function

$$\mathcal{Z}_n = \int_{\mathcal{N}_n} e^{-2n \operatorname{trace} Q(M)} dM_n,$$

where dM_n is the Riemannian volume form on \mathcal{N}_n inherited from the space \mathbb{C}^{n^2} of all $n \times n$ matrices, and where $\operatorname{trace} Q : \mathcal{N}_n \rightarrow \mathbb{R} \cup \{+\infty\}$ is the random variable

$$\operatorname{trace} Q(M) = \sum_{\lambda_j \in \operatorname{spec}(M)} Q(\lambda_j),$$

i.e., the usual trace of the matrix $Q(M)$. We equip \mathcal{N}_n with the probability measure

$$d\mathcal{P}_n = \frac{1}{\mathcal{Z}_n} e^{-2n \operatorname{trace} Q(M)} dM_n,$$

and speak of the *random normal matrix ensemble* or "RNM-ensemble" associated with Q .

The measure \mathcal{P}_n induces a measure \mathbf{P}_n on the space \mathbb{C}^n of eigenvalues, which is known as the *density of states* in the external field Q ; it is given by

$$d\mathbf{P}_n(\lambda) = \frac{1}{Z_n} e^{-H_n(\lambda)} dA_n(\lambda), \quad \lambda = (\lambda_j)_1^n \in \mathbb{C}^n.$$

Here we have put

$$H_n(\lambda) = \sum_{j \neq k} \log \frac{1}{|\lambda_j - \lambda_k|} + 2n \sum_{j=1}^n Q(\lambda_j),$$

and $dA_n(\lambda) = d^2\lambda_1 \cdots d^2\lambda_n$ denotes Lebesgue measure in \mathbb{C}^n , while Z_n is the normalizing constant giving \mathbf{P}_n unit mass. By a slight abuse of language, we will refer to Z_n as the partition function of the ensemble.

Notice that H_n is the energy (Hamiltonian) of a system of n identical point charges in the plane located at the points λ_j , under influence of the external field $2nQ$. In this interpretation, \mathbf{P}_n is the law of the Coulomb gas in the external magnetic field $2nQ$ (at inverse temperature $\beta = 2$). In particular, this explains the repelling nature of the eigenvalues of random normal matrices; they tend to be very spread out in the vicinity of the droplet, just like point charges would.

Consider the n -point configuration ("set" with possible repeated elements) $\{\lambda_j\}_1^n$ of eigenvalues of a normal matrix picked randomly with respect to \mathcal{P}_n . In an obvious manner, the measure \mathbf{P}_n induces a probability law on the n -point configuration space; this is the law of the n -point process $\Psi_n = \{\lambda_j\}_1^n$ associated to Q .

It is well-known that the process Ψ_n is *determinantal*. This means that there exists a Hermitian function \mathbf{K}_n , called the *correlation kernel of the process* such that the density of states can be

represented in the form

$$d\mathbf{P}_n(\lambda) = \frac{1}{n!} \det \left(\mathbf{K}_n(\lambda_j, \lambda_k) \right)_{j,k=1}^n dA_n(\lambda), \quad \lambda \in \mathbb{C}^n.$$

One has

$$\mathbf{K}_n(z, w) = K_n(z, w) e^{-n(Q(z) + Q(w))},$$

where K_n is the reproducing kernel of the space $\mathcal{P}_n(e^{-2nQ})$ of analytic polynomials of degree at most $n - 1$ with norm induced from the usual L^2 space on \mathbb{C} associated with the weight function e^{-2nQ} . Alternatively, we can regard \mathbf{K}_n as the reproducing kernel for the subspace

$$W_n = \{pe^{-nQ}; p \text{ is an analytic polynomial of degree less than } n\} \subset L^2(\mathbb{C}).$$

We have the frequently useful identities

$$f(z) = \int_{\mathbb{C}} f(w) \overline{\mathbf{K}_n(z, w)} d^2w, \quad f \in W_n,$$

and

$$\int_{\mathbb{C}} \mathbf{K}_n(z, z) d^2z = n.$$

Moreover, if \mathbf{E}_n denotes expectation with respect to \mathbf{P}_n , and if f is a function depending on k variables, where $k \leq n$, then

$$\mathbf{E}_n[f(\lambda_1, \dots, \lambda_k)] = \frac{(n-k)!k!}{n!} \mathbf{E}_k[f(\lambda_1, \dots, \lambda_k)].$$

We refer to [7], [18], [9], [10], [3], [14] for more details on point-processes and random matrices.

1.2. The equilibrium measure and the droplet. We are interested in the asymptotic distribution of eigenvalues as n , the size of the matrices, increases indefinitely. Let u_n denote the one-point function of \mathbf{P}_n , i.e.,

$$u_n(\lambda) = \frac{1}{n} \mathbf{K}_n(\lambda, \lambda), \quad \lambda \in \mathbb{C}.$$

With a suitable function f on \mathbb{C} , we associate the random variable $\text{Tr}_n[f]$ on the probability space $(\mathbb{C}^n, \mathbf{P}_n)$ via

$$\text{Tr}_n[f](\lambda) = \sum_{i=1}^n f(\lambda_i).$$

The expectation is given by

$$\mathbf{E}_n(\text{Tr}_n[f]) = n \int_{\mathbb{C}} f \cdot u_n.$$

According to Johansson (see [10]) we have weak-star convergence of the measures

$$d\sigma_n(z) = u_n(z) d^2z$$

to some probability measure $\sigma = \sigma(Q)$ on \mathbb{C} .

In fact, σ is the Frostman equilibrium measure of the logarithmic potential theory with external field Q . We briefly recall the definition and some basic properties of this probability measure, cf. [16] and [10] for proofs and further details.

Let $S = \text{supp } \sigma$ and assume that Q is C^2 -smooth in some neighbourhood of S . Then S is compact, Q is subharmonic on S , and σ is absolutely continuous with density

$$u = \frac{1}{2\pi} \Delta Q \cdot \mathbf{1}_S.$$

We refer to the compact set $S = S_Q$ as the *droplet* corresponding to the external field Q .

Our present goal is to describe the fluctuations of the density field $\mu_n = \sum_{j=1}^n \delta_{\lambda_j}$ around the equilibrium. More precisely, we will study the distribution (linear statistic)

$$f \mapsto \mu_n(f) - n\sigma(f) = \text{Tr}_n[f] - n\sigma(f), \quad f \in C_0^\infty(\mathbb{C}).$$

We will denote by ν_n the measure with density $n(u_n - u)$, i.e.,

$$\nu_n[f] = \mathbf{E}_n[\text{Tr}_n[f]] - n\sigma(f) = n(\sigma_n - \sigma)(f), \quad f \in C_0^\infty(\mathbb{C}).$$

1.3. Assumptions on the potential. To state the main results of the paper we make the following three assumptions:

(A1) (smoothness) Q is real analytic (written $Q \in C^\omega$) in some neighborhood of the droplet $S = S_Q$;

(A2) (regularity) $\Delta Q \neq 0$ in S ;

(A3) (topology) ∂S is a C^ω -smooth Jordan curve.

We will comment on the nature and consequences of these assumptions later. Let us denote

$$L = \log \Delta Q.$$

This function is well-defined and C^ω in a neighborhood of the droplet.

1.4. The Neumann jump operator. We will use the following general system of notation. If g is a continuous function defined in a neighborhood of S , then we write g^S for the function on the Riemann sphere $\hat{\mathbb{C}}$ such that g^S equals g in S while g^S equals the harmonic extension of $g|_{\partial S}$ to $\hat{\mathbb{C}} \setminus S$ on that set.

If g is smooth on S , then

$$\mathcal{N}_\Omega g := -\frac{\partial g|_S}{\partial n}, \quad \Omega := \text{int}(S),$$

where n is the (exterior) unit normal of Ω . We define the normal derivative $\mathcal{N}_\Omega g$ for the complementary domain $\Omega_* := \hat{\mathbb{C}} \setminus S$ similarly. If both normal derivatives exist, then we define (Neumann's jump)

$$\mathcal{N}g \equiv \mathcal{N}_{\partial S} := \mathcal{N}_\Omega g + \mathcal{N}_{\Omega_*} g.$$

By Green's formula we have the identity (of measures)

$$(1.2) \quad \Delta g^S = \Delta g \cdot 1_\Omega + \mathcal{N}(g^S) ds,$$

where ds is the arclength measure on ∂S .

We now verify (1.2). Let ϕ be a test function. The left hand side in (1.2) applied to ϕ is

$$\int_{\mathbb{C}} \phi \Delta g^S = \int_{\mathbb{C}} g^S \Delta \phi = \int_S g \Delta \phi + \int_{\mathbb{C} \setminus S} g^S \Delta \phi,$$

and the right hand side is

$$\int_S \phi \Delta g + \int \phi \mathcal{N}(g^S) ds.$$

Thus we need to check that

$$\int_S (g \Delta \phi - \phi \Delta g) + \int_{\mathbb{C} \setminus S} (g^S \Delta \phi - \phi \Delta g^S) = \int \phi \mathcal{N}(g^S) ds.$$

But the expression in the left hand side is

$$\int (g\partial_n\phi - \phi\partial_n g)ds + \int (g\partial_{n^*}\phi - \phi\partial_{n^*}g^S)ds = - \int (\phi\partial_n g + \phi\partial_{n^*}g^S)ds = \int \phi \mathcal{N}(g^S)ds,$$

and (1.2) is proved.

1.5. Main results. We have the following results.

Theorem 1.1. *For all test functions $f \in C_0^\infty(\mathbb{C})$, the limit*

$$v(f) := \lim_{n \rightarrow \infty} v_n(f)$$

exists, and

$$v(f) = \frac{1}{8\pi} \left[\int_S \Delta f + \int_S f \Delta L + \int_{\partial S} f \mathcal{N}(L^S) ds \right].$$

Equivalently, we have

$$v_n \rightarrow v = \frac{1}{8\pi} \Delta (1_S + L^S)$$

in the sense of distributions.

Theorem 1.2. *Let $h \in C_0^\infty(\mathbb{C})$ be a real-valued test function. Then, as $n \rightarrow \infty$,*

$$\text{trace}_n h - \mathbf{E}_n \text{trace}_n h \rightarrow N\left(0, \frac{1}{2\pi} \int_{\mathbb{C}} |\nabla h^S|^2\right).$$

The last statement means convergence in distribution of random variables to a normal law with indicated expectation and variance. As noted in [3], Section 7, the result can be restated in terms of convergence of random fields to a Gaussian field on S with free boundary conditions.

1.6. Derivation of Theorem 1.2. We now show, using the variational approach due to Johansson [13], that the Gaussian convergence stated in Theorem 1.2 follows from a generalized version of Theorem 1.1, which we now state.

Fix a real-valued test function h and consider the perturbed potentials

$$\tilde{Q}_n := Q - \frac{1}{n}h.$$

We denote by \tilde{u}_n the one-point function of the density of states $\tilde{\mathbf{P}}_n$ associated with the potential \tilde{Q}_n . We write $\tilde{\sigma}_n$ for the measure with density \tilde{u}_n and \tilde{v}_n for the measure $n(\tilde{\sigma}_n - \sigma)$, i.e.,

$$(1.3) \quad \tilde{v}_n[f] = n\tilde{\sigma}_n(f) - n\sigma(f) = \tilde{\mathbf{E}}_n \text{Tr}_n[f] - n\sigma(f).$$

Theorem 1.3. *For all $f \in C_0^\infty(\mathbb{C})$ we have*

$$\tilde{v}_n(f) - v_n(f) \rightarrow \frac{1}{2\pi} \int_{\mathbb{C}} \nabla f^S \cdot \nabla h^S.$$

A proof of Theorem 1.3 is given in Section 4.

Claim. Theorem 1.2 is a consequence of Theorem 1.3.

Proof. Denote $X_n = \text{Tr}_n h - \mathbf{E}_n \text{Tr}_n h$ and write $a_n = \tilde{\mathbf{E}}_n X_n$. By Theorem 1.3,

$$a_n \rightarrow a \quad \text{where} \quad a = \frac{1}{2\pi} \int_{\mathbb{C}} \nabla h^S \cdot \nabla h^S.$$

More generally, let $\lambda \geq 0$ be a parameter, and let $\tilde{\mathbf{E}}_{n,\lambda}$ denote expectation corresponding to the potential $Q - (\lambda h)/n$. Write

$$F_n(\lambda) := \log \mathbf{E}_n e^{\lambda X_n}, \quad 0 \leq \lambda \leq 1.$$

Since $F'_n(\lambda) = \tilde{\mathbf{E}}_{n,\lambda} X_n$, Theorem 1.3 implies

$$(1.4) \quad F'_n(\lambda) = \tilde{\mathbf{E}}_{n,\lambda} X_n \rightarrow \lambda a,$$

and

$$(1.5) \quad \log \mathbf{E}_n e^{X_n} = F_n(1) = \int_0^1 F'_n(\lambda) d\lambda \rightarrow \frac{a}{2}, \quad \text{as } n \rightarrow \infty.$$

Here we use the convexity of the functions F_n ,

$$F''_n(\lambda) = \tilde{\mathbf{E}}_{n,\lambda} X_n^2 - (\tilde{\mathbf{E}}_{n,\lambda} X_n)^2 \geq 0,$$

which implies that the convergence in (1.4) is dominated:

$$0 = F'_n(0) \leq F'_n(\lambda) \leq F'_n(1).$$

Replacing h by th where $t \in \mathbb{R}$, we get $\mathbf{E}_n(e^{tX_n}) \rightarrow e^{t^2 a/2}$ as $n \rightarrow \infty$, i.e., we have convergence of all moments of X_n to the moments of the normal $N(0, a)$ distribution. It is well known that this implies convergence in distribution, viz. Theorem 1.2 follows. \square

1.7. Comments.

(a) *Related Work.* The one-dimensional analog of the weighted RNM theory is the more well-known random Hermitian matrix theory, which was studied by Johansson in the important paper [13]. Indeed, Johansson obtains results not only random Hermitian matrix ensembles, but for more general (one-dimensional) β -ensembles. The paper [13] was one of our main sources of inspiration for the present work.

In [3], it was shown that the convergence in theorems 1.1 and 1.2 holds for test functions supported in the interior of the droplet. See also [6]. In [3], we also announced theorems 1.1 and 1.2 and proved several consequences of them, e.g. the convergence of Berezin measures, rooted at a point in the exterior of S , to a harmonic measure.

Rider and Virág [15] proved theorems 1.1 and 1.2 in the special case $Q(z) = |z|^2$ (the *Ginibre ensemble*). The paper [8] contains results in this direction for β -Ginibre ensembles for some special values of β .

Our main technique, the method of Ward identities, is common practice in field theories. In this method, one uses reparametrization invariance of the partition function to deduce exact relations satisfied by the joint intensity functions of the ensemble. In particular, the method was applied on the physical level by Wiegmann, Zabrodin et al. to study RNM ensembles as well as more general OCP ensembles. See e.g. the papers [19], [20], [21], [22]. A one-dimensional version of Ward's identity was also used by Johansson in [13].

Finally, we wish to mention that one of the topics in this paper, the behaviour of fluctuations near the boundary, is analyzed from another perspective in the forthcoming paper [4].

(b) *Assumptions on the potential.* We here comment on the assumptions (A1)–(A3) which we require of the potential Q .

The C^ω assumption (A1) is natural for the study of fluctuation properties near the boundary of the droplet. (For test functions supported in the interior, one can do with less regularity.)

Using Sakai's theory [18], it can be shown that conditions (A1) and (A2) imply that ∂S is a union of finitely many C^ω curves with a finite number of singularities of known types. It is not difficult to complete a proof using arguments from [11], Section 4.

We rule out singularities by the regularity assumption in (A3). What happens in the presence of singularities is probably an interesting topic, which we have not approached.

Without singularities the boundary of the droplet is a union of finitely many C^ω Jordan curves. Assumption (A3) means that we only consider the case of a single boundary component. Our methods extend without difficulty to the case of a multiply connected droplet. The disconnected case requires further analysis, and is not considered in this paper.

(c) *Droplets and potential theory.* We here state the properties of droplets that will be needed for our analysis. Proofs for these properties can be found in [16] and [10].

We will write \check{Q} for the maximal subharmonic function $\leq Q$ which grows as $\log |z| + O(1)$ when $|z| \rightarrow \infty$. We have that $\check{Q} = Q$ on S while \check{Q} is $C^{1,1}$ -smooth on \mathbb{C} and

$$\check{Q}(z) = Q^S(z) + G(z, \infty), \quad z \in \mathbb{C} \setminus S,$$

where G is the classical Green's function of $\mathbb{C} \setminus S$. In particular, if

$$U^\sigma(z) = \int \log \frac{1}{|z - \zeta|} d\sigma(\zeta)$$

denotes the logarithmic potential of the equilibrium measure, then

$$(1.6) \quad \check{Q} + U^\sigma \equiv \text{const.}$$

The following proposition sums up some basic properties of the droplet and the function \check{Q} .

Proposition 1.4. *Suppose Q satisfies (A1)–(A3). Then ∂S is a C^ω Jordan curve, $\check{Q} \in W^{2,\infty}(\mathbb{C})$, and therefore*

$$\partial \check{Q} = (\partial Q)^S.$$

Furthermore, we have

$$(1.7) \quad Q(z) - \check{Q}(z) \asymp \delta(z)^2, \quad z \notin S, \quad \delta(z) \rightarrow 0,$$

where $\delta(z)$ denotes the distance from z to the droplet.

(d) *Joint intensities.* We will occasionally use the intensity k -point function of the process Ψ_n . This is the function defined by

$$R_n^{(k)}(z_1, \dots, z_k) = \lim_{\varepsilon \rightarrow 0} \frac{\mathbf{P}_n \left(\bigcap_{j=1}^k \{ \Psi_n \cap D(z_j, \varepsilon) \neq \emptyset \} \right)}{\pi^k \varepsilon^{2k}} = \det \left(\mathbf{K}_n(z_i, z_j) \right)_{i,j=1}^k.$$

In particular, $R_n^{(1)} = n u_n$.

(e) *Organization of the paper.* We will derive the following statement which combines theorems 1.1 and 1.3 (whence, by Lemma 1.6, it implies Theorem 1.2).

Main formula: Let \tilde{v}_n be the measure defined in (1.3). Then

$$(1.8) \quad \lim_{n \rightarrow \infty} \tilde{v}_n(f) = \frac{1}{8\pi} \left[\int_S \Delta f + \int_S f \Delta L + \int_{\partial S} f \mathcal{N}(L^S) ds \right] + \frac{1}{2\pi} \int_{\mathbb{C}} \nabla f^S \cdot \nabla h^S.$$

Our proof of this formula is based on the limit form of Ward's identities which we discuss in the next section. To justify this limit form we need to estimate certain error terms; this is done in Section 3. In the proof, we refer to some basic estimates of polynomial Bergman kernels, which we collect in the appendix. The proof of the main theorem is completed in Section 4.

2. WARD IDENTITIES

2.1. Exact identities. For a suitable function v on \mathbb{C} we define a random variable $W_n^+[v]$ on the probability space $(\mathbb{C}^n, \mathbf{P}_n)$ by

$$W_n^+[v] = \frac{1}{2} \sum_{j \neq k} \frac{v(\lambda_j) - v(\lambda_k)}{\lambda_j - \lambda_k} - 2n \operatorname{Tr}_n[v \partial Q] + \operatorname{Tr}_n[\partial v].$$

Proposition 2.1. *Let $v : \mathbb{C} \rightarrow \mathbb{C}$ be Lipschitz continuous with compact support. Then*

$$\mathbf{E}_n W_n^+[v] = 0.$$

Proof. We write

$$W_n^+[v] = I_n[v] - II_n[v] + III_n[v]$$

where (almost everywhere)

$$I_n[v](z) = \frac{1}{2} \sum_{j \neq k}^n \frac{v(z_j) - v(z_k)}{z_j - z_k} \quad ; \quad II_n[v](z) = 2 \sum_{j=1}^n \partial Q(z_j) v(z_j) \quad ; \quad III_n[v](z) = \sum_{j=1}^n \partial v(z_j).$$

Let ε be a real parameter and put $z_j = \phi(\zeta_j) = \zeta_j + \varepsilon v(\zeta_j)/2$, $1 \leq j \leq n$. Then

$$d^2 z_j = \left(\left| \partial \phi(\zeta_j) \right|^2 + \left| \bar{\partial} \phi(\zeta_j) \right|^2 \right) d^2 \zeta_j = \left[1 + \varepsilon \operatorname{Re} \partial v(\zeta_j) + O(\varepsilon^2) \right] d^2 \zeta_j,$$

so that (with $III_n = III_n[v]$)

$$dA_n(z) = \left[1 + \varepsilon \operatorname{Re} III_n(\zeta) + O(\varepsilon^2) \right] dA_n(\zeta).$$

Moreover,

$$\begin{aligned} \log |z_i - z_j|^2 &= \log |\zeta_i - \zeta_j|^2 + \log \left| 1 + \frac{\varepsilon v(\zeta_i) - v(\zeta_j)}{\zeta_i - \zeta_j} \right|^2 = \\ &= \log |\zeta_i - \zeta_j|^2 + \varepsilon \operatorname{Re} \frac{v(\zeta_i) - v(\zeta_j)}{\zeta_i - \zeta_j} + O(\varepsilon^2), \end{aligned}$$

so that

$$(2.1) \quad \sum_{j \neq k}^n \log |z_j - z_k|^{-1} = \sum_{j \neq k}^n \log |\zeta_j - \zeta_k|^{-1} - \varepsilon \operatorname{Re} I_n(\zeta) + O(\varepsilon^2), \quad \text{as } \varepsilon \rightarrow 0.$$

Finally,

$$Q(z_j) = Q\left(\zeta_j + \frac{\varepsilon}{2}v(\zeta_j)\right) = Q(\zeta_j) + \varepsilon \operatorname{Re} \left(\partial Q(\zeta_j) v(\zeta_j) \right),$$

so

$$(2.2) \quad 2n \sum_{j=1}^n Q(z_j) = 2n \sum_{j=1}^n Q(\zeta_j) + \varepsilon \operatorname{Re} II_n(\zeta) + O(\varepsilon^2).$$

Now (2.1) and (2.2) imply that the Hamiltonian $H_n(z) = \sum_{j \neq k} \log |z_j - z_k|^{-1} + 2n \sum_{j=1}^n Q(z_j)$ satisfies

$$(2.3) \quad H_n(z) = H_n(\zeta) + \varepsilon \cdot \operatorname{Re} (-I_n(\zeta) + II_n(\zeta)) + O(\varepsilon^2).$$

It follows that

$$Z_n := \int_{\mathbb{C}^n} e^{-H_n(z)} dA_n(z) = \int_{\mathbb{C}^n} e^{-H_n(\zeta) - \varepsilon \operatorname{Re} (-I_n(\zeta) + II_n(\zeta)) + O(\varepsilon^2)} \left[1 + \varepsilon \operatorname{Re} III_n(\zeta) + O(\varepsilon^2) \right] dA_n(\zeta).$$

Since the integral is independent of ε , the coefficient of ε in the right hand side must vanish, which means that

$$(2.4) \quad \operatorname{Re} \int_{\mathbb{C}^n} (III_n(\zeta) + I_n(\zeta) - II_n(\zeta)) e^{-H_n(\zeta)} dA_n(\zeta) = 0,$$

or $\operatorname{Re} \mathbf{E}_n W_n^+[v] = 0$. Replacing v by iv in the preceding argument gives $\operatorname{Im} \mathbf{E}_n W_n^+[v] = 0$ and the proposition follows. \square

Applying Proposition 2.1 to the potential $\tilde{Q}_n = Q - h/n$, we get the identity

$$(2.5) \quad \tilde{\mathbf{E}}_n \tilde{W}_n^+[v] = 0,$$

where

$$(2.6) \quad \tilde{W}_n^+[v] = W_n^+[v] + 2 \operatorname{Tr}_n[v \partial h].$$

If we denote

$$B_n[v] = \frac{1}{2n} \sum_{i \neq j} \frac{v(\lambda_i) - v(\lambda_j)}{\lambda_i - \lambda_j},$$

we can rewrite (2.5) and (2.6) as follows,

$$(2.7) \quad \tilde{\mathbf{E}}_n B_n[v] = 2 \tilde{\mathbf{E}}_n \operatorname{Tr}_n[v \partial Q] - \tilde{\sigma}_n(\partial v + 2v \partial h),$$

where we recall that $\tilde{\sigma}_n$ is the measure with density \tilde{u}_n .

2.2. Cauchy kernels. For each $z \in \mathbb{C}$ let k_z denote the function

$$k_z(\lambda) = \frac{1}{z - \lambda},$$

so $z \mapsto \sigma(k_z)$ is the Cauchy transform of the the measure σ . We have (see (1.6))

$$\sigma(k_z) = 2\partial\check{Q}(z).$$

We will also consider the Cauchy integrals $\sigma_n(k_z)$ and $\tilde{\sigma}_n(k_z)$. We have

$$\bar{\partial}_z[\sigma_n(k_z)] = \pi u_n(z), \quad \bar{\partial}_z[\tilde{\sigma}_n(k_z)] = \pi \tilde{u}_n(z), \quad z \in \mathbb{C},$$

and

$$\tilde{\sigma}_n(k_z) \rightarrow \sigma(k_z)$$

with uniform convergence on \mathbb{C} (the uniform convergence follows easily from the one-point function estimates in Lemma 3.1 and Theorem 3.2).

Let us now introduce the functions

$$D_n(z) = v_n(k_z) \quad ; \quad \tilde{D}_n(z) = \tilde{v}_n(k_z).$$

We have

$$(2.8) \quad \tilde{D}_n(z) = n[\tilde{\sigma}_n(k_z) - 2\partial\check{Q}(z)] \quad , \quad \bar{\partial}\tilde{D}_n = n\pi(\tilde{u}_n - u),$$

and if f is a test function, then

$$(2.9) \quad \tilde{v}_n(f) = \frac{1}{\pi} \int f \bar{\partial}\tilde{D}_n = -\frac{1}{\pi} \int \bar{\partial}f \cdot \tilde{D}_n.$$

Let $\tilde{\mathbf{K}}_n$ denote the correlation kernel with respect to \tilde{Q}_n . Using \tilde{D}_n , we can rewrite the $B_n[v]$ term in the Ward identity as follows.

Lemma 2.2. *One has that*

$$\tilde{\mathbf{E}}_n B_n[v] = 2 \int v \cdot \partial\check{Q} \cdot \tilde{\mathbf{K}}_n + \int v \tilde{D}_n \tilde{u}_n - \frac{1}{2n} \iint \frac{v(z) - v(w)}{z - w} |\tilde{\mathbf{K}}_n(z, w)|^2.$$

(In the first integral $\tilde{\mathbf{K}}_n(z)$ means the 1-point intensity $\tilde{R}_n^{(1)}(z) = \tilde{\mathbf{K}}_n(z, z)$.)

Proof. We have

$$\tilde{\mathbf{E}}_n B_n[v] = \frac{1}{2n} \iint_{\mathbb{C}^2} \frac{v(z) - v(w)}{z - w} \tilde{R}_n^{(2)}(z, w),$$

where

$$\tilde{R}_n^{(2)}(z, w) = \tilde{\mathbf{K}}_n(z) \tilde{\mathbf{K}}_n(w) - |\tilde{\mathbf{K}}_n(z, w)|^2.$$

The integral involving $\tilde{\mathbf{K}}_n(z) \tilde{\mathbf{K}}_n(w)$ is

$$\frac{1}{n} \iint_{\mathbb{C}^2} \frac{v(z)}{z - w} \tilde{\mathbf{K}}_n(z) \tilde{\mathbf{K}}_n(w) = \int_{\mathbb{C}} v(z) \cdot \tilde{\mathbf{K}}_n(z) \cdot \tilde{\sigma}_n(k_z),$$

and by (2.8) $\tilde{\sigma}_n(k_z) = \frac{1}{n} \tilde{D}_n + 2\partial\check{Q}$. □

2.3. Limit form of Ward's identity. The main formula (1.8) will be derived from Theorem 2.3 below. In this theorem we make the following assumptions on the vector field v :

- (i) v is compactly supported;
- (ii) v is Lip-continuous in \mathbb{C} ;
- (iii) v is uniformly C^2 -smooth in $\mathbb{C} \setminus \partial S$.

(The last condition means that the restriction of v to S and the restriction to $(\mathbb{C} \setminus S) \cup \partial S$ are both C^2 -smooth.)

Theorem 2.3. *If v satisfies (i)-(iii), then as $n \rightarrow \infty$,*

$$\frac{2}{\pi} \int_S v \tilde{D}_n \partial\bar{\partial}Q + \frac{2}{\pi} \int_{\mathbb{C} \setminus S} v(\partial\check{Q} - \partial Q) \bar{\partial}\tilde{D}_n \rightarrow -\frac{1}{2} \sigma(\partial v) - 2\sigma(v\partial h).$$

Before we come to the proof, we check that it is possible to integrate by parts in the second integral in Theorem 2.3. To control the boundary term we can use the next lemma.

Lemma 2.4. *For every fixed n we have*

$$|\tilde{D}_n(z)| \lesssim \frac{1}{|z|^2}, \quad (z \rightarrow \infty).$$

Proof. We have

$$\left| \frac{\tilde{D}_n(z)}{n} \right| = \left| \int \frac{(\tilde{u}_n - u)d^2\lambda}{z - \lambda} \right| = \left| \int \left[\frac{1}{z - \lambda} - \frac{1}{z} \right] (\tilde{u}_n - u)d^2\lambda \right|$$

Since

$$\frac{1}{z - \lambda} - \frac{1}{z} = \frac{1}{z^2} \frac{\lambda}{1 - \lambda/z},$$

we need to show that the integrals

$$\int \frac{|\lambda| |\tilde{u}_n - u|}{|1 - \lambda/z|} d^2\lambda$$

are uniformly bounded. To prove this, we only need the estimate $\tilde{u}_n(\lambda) \lesssim \frac{1}{|\lambda|^3}$, which holds (for sufficiently large n) by the growth assumption (1.2) and the simple estimate $\tilde{u}_n(\lambda) \leq C \exp(-2n(Q(\lambda) - \check{Q}(\lambda)))$, which is given below in Lemma 3.1. \square

Using that $\partial Q = \partial \check{Q}$ in the interior of S , we deduce the following corollary of Theorem 2.3.

Corollary 2.5. ("Limit Ward identity") *Suppose that v satisfies conditions (i)-(iii). Then as $n \rightarrow \infty$ we have the convergence*

$$\frac{2}{\pi} \int_{\mathbb{C}} [v \partial \bar{\partial} Q + \bar{\partial} v (\partial Q - \partial \check{Q})] \tilde{D}_n \rightarrow -\frac{1}{2} \sigma(\partial v) - 2\sigma(v \partial h).$$

2.4. Error terms and the proof of Theorem 2.3. Theorem 2.3 follows if we combine the expressions for $\tilde{\mathbf{E}}_n B_n[v]$ in (2.7) and Lemma 2.2 and use the following approximations of the last two terms in Lemma 2.2. More precisely, if we introduce the *first error term* by

$$(2.10) \quad \frac{1}{n} \iint \frac{v(z) - v(w)}{z - w} |\tilde{\mathbf{K}}_n(z, w)|^2 = \tilde{\sigma}_n(\partial v) + \varepsilon_n^1[v],$$

and the *second error term* by

$$(2.11) \quad \varepsilon_n^2[v] = \pi \int v \tilde{D}_n (\tilde{u}_n - u) = -\frac{1}{2} \int \bar{\partial} v \frac{\tilde{D}_n^2}{n}.$$

Using (2.7), Lemma 2.2, and that $\partial \check{Q} = \partial Q$ a.e. on S , one deduces that

$$(2.12) \quad \frac{2}{\pi} \int_S v \tilde{D}_n \partial \bar{\partial} Q + \frac{2}{\pi} \int_{\mathbb{C} \setminus S} v (\partial \check{Q} - \partial Q) \bar{\partial} \tilde{D}_n = -\frac{1}{2} \sigma(\partial v) - 2\sigma(v \partial h) + \frac{1}{2} \varepsilon_n^1[v] - \frac{1}{\pi} \varepsilon_n^2[v] + o(1),$$

where $o(1) = (\sigma - \tilde{\sigma}_n)(\partial v/2 + 2v \partial h)$ converges to zero as $n \rightarrow \infty$ by the one-point function estimates in Lemma 3.1 and Theorem 3.2.

In the next section we will show that for each v satisfying conditions (i)-(iii), the error terms $\varepsilon_n^j[v]$ tend to zero as $n \rightarrow \infty$, which will finish the proof of Theorem 2.3.

3. ESTIMATES OF THE ERROR TERMS

3.1. Estimates of the kernel $\tilde{\mathbf{K}}_n$. We will use two different estimates for the correlation kernel, one to handle the interior and another for the exterior of the droplet.

(a) *Exterior estimate.* Recall that $\tilde{\mathbf{K}}_n(z, w)$ is the kernel of the n -point process associated with potential $\tilde{Q}_n = Q - h/n$; as usual, we write $\tilde{\mathbf{K}}_n(z) = \tilde{\mathbf{K}}_n(z, z)$. We have the following global estimate, which is particularly useful in the exterior of the droplet.

Lemma 3.1. *For all $z \in \mathbb{C}$ we have*

$$\tilde{\mathbf{K}}_n(z) \lesssim ne^{-2n(Q-\tilde{Q})(z)},$$

where the constant is independent of n and z .

This estimate has been recorded (see e.g. [2], Section 3) for the kernels \mathbf{K}_n , i.e. in the case $h = 0$. Since obviously

$$\|p\|_{e^{-2n\tilde{Q}_n}} \asymp \|p\|_{e^{-2nQ}},$$

we have $\tilde{\mathbf{K}}_n(z) \asymp \mathbf{K}_n(z)$ with a constant independent of z . Indeed, $\mathbf{K}_n(z)$ is the supremum of $|p(z)|^2 e^{-2nQ(z)}$ where p is an analytic polynomial of degree less than n such that $\|p\|_{e^{-2nQ}} \leq 1$, and we have an analogous supremum characterization of $\tilde{\mathbf{K}}_n(z)$. Hence the case $h \neq 0$ does not require any special treatment.

In the following we write

$$\delta(z) = \text{dist}(z, \partial S)$$

and

$$\delta_n = \frac{\log^2 n}{\sqrt{n}}.$$

By our assumption on the droplet (see Proposition 1.4) we have

$$Q(z) - \tilde{Q}(z) \gtrsim \delta^2(z), \quad z \notin S, \quad \delta(z) \rightarrow 0.$$

In view of the growth assumption (1.1), it follows that for any $N > 0$ there exists C_N such that $\tilde{\mathbf{K}}_n(z) \lesssim C_N n^{-N}$ when z is outside the δ_n -neighborhood of S .

(b) *Interior estimate.* Recall that we assume that Q is real analytic in some neighbourhood of S . This means that we can extend Q to a complex analytic function of two variables in some neighbourhood in \mathbb{C}^2 of the anti-diagonal

$$\{(z, \bar{z}) : z \in S\} \subset \mathbb{C}^2.$$

We will use the same letter Q for this extension, so

$$Q(z) = Q(z, \bar{z}).$$

We have

$$Q(z, w) = \overline{Q(\bar{w}, \bar{z})}$$

and

$$\partial_1 Q(z, \bar{z}) = \partial Q(z), \quad \partial_1 \partial_2 Q(z, \bar{z}) = \partial \bar{\partial} Q(z), \quad \partial_1^2 Q(z, \bar{z}) = \partial^2 Q(z), \quad \text{etc.}$$

With the help of this extension, one can show that the leading contribution to the kernel \mathbf{K}_n is of the form

$$(3.1) \quad \mathbf{K}_n^\#(z, w) = \frac{2}{\pi} (\partial_1 \partial_2 Q)(z, \bar{w}) n e^{n[2Q(z, \bar{w}) - Q(z) - Q(w)]}.$$

In particular, we have

$$\mathbf{K}_n^\#(w, w) = \frac{n \Delta Q(w)}{2\pi}, \quad (w \in S).$$

We shall use the following estimate in the interior.

Theorem 3.2. *If $z \in S$, $\delta(z) > 2\delta_n$, and if $|z - w| < \delta_n$, then*

$$|\tilde{\mathbf{K}}_n(z, w)| = |\mathbf{K}_n^\#(z, w)| + O(1),$$

where the constant in $O(1)$ depend on Q and h but not on n .

Similar types of expansions are discussed e.g. in [5], [1], [2]. As there is no convenient reference for this particular result, and to make the paper selfcontained, we include a proof in the appendix.

We now turn to the proof that the error terms $\varepsilon_n^1[v]$ and $\varepsilon_n^2[v]$ are negligible. See (2.10) and (2.11). Our proof uses only the estimates of the kernels $\tilde{\mathbf{K}}_n$ mentioned above. Since the form of these estimates is the same for all perturbation functions h , we can without loss of generality set $h = 0$, which will simplify our notation – no need to put tildes on numerous letters.

3.2. First error term. We start with the observation that if $w \in S$ and $\delta(w) > 2\delta_n$ then at short distances the so called Berezin kernel rooted at w

$$B_n^{(w)}(z) = \frac{|\mathbf{K}_n(z, w)|^2}{\mathbf{K}_n(w, w)}$$

is close to the heat kernel

$$H_n^{(w)}(z) = \frac{1}{\pi} c n e^{-c n |z - w|^2}, \quad c := 2\partial\bar{\partial}Q(w).$$

Both kernels determine probability measures indexed by w . Most of the heat kernel measure is concentrated in the disc $D(w, \delta_n)$,

$$\int_{\mathbb{C} \setminus D(w, \delta_n)} H_n^{(w)}(z) dA(z) \lesssim \frac{1}{n^N},$$

where N denotes an arbitrary (large) positive number.

Lemma 3.3. *Suppose that $w \in S$, $\delta(w) > 2\delta_n$ and $|z - w| < \delta_n$. Then*

$$|B_n^{(w)}(z) - H_n^{(w)}(z)| \lesssim n\delta_n.$$

Proof. By Theorem 3.2 we have

$$B_n^{(w)}(z) = \frac{|\mathbf{K}_n^\#(z, w)|^2}{\mathbf{K}_n^\#(w, w)} + O(1).$$

Next, we fix w and apply Taylor's formula to the function $z \mapsto \mathbf{K}^\#(z, w)$ at $z = w$. Using the explicit formula (3.1) for this function, and that

$$\begin{aligned} Q(z, \bar{w}) + \overline{Q(z, \bar{w})} - Q(z, \bar{z}) - Q(w, \bar{w}) &= [Q(z, \bar{w}) - Q(w, \bar{w})] + [Q(w, \bar{z}) - Q(z, \bar{z})] = \\ &= \partial Q(w)(z - w) + \frac{1}{2} \partial^2 Q(w)(z - w)^2 + \partial Q(z)(w - z) + \frac{1}{2} \partial^2 Q(z)(z - w)^2 + \dots = \\ &= [\partial Q(w) - \partial Q(z)](z - w) + \partial^2 Q(w)(z - w)^2 + \dots = -\partial \bar{\partial} Q(w)|z - w|^2 + \dots \end{aligned}$$

we get

$$\begin{aligned} \frac{|\mathbf{K}_n^\#(z, w)|^2}{\mathbf{K}_n^\#(w, w)} &= \frac{1}{\pi} [(c + O(|z - w|))] n e^{-cn|z - w|^2 + O(n|z - w|^3)} \\ &= H_n^{\langle w \rangle}(z) + O(n|z - w|), \end{aligned}$$

and the assertion follows. \square

Corollary 3.4. *If $w \in S$ and $\delta(w) > 2\delta_n$, then*

$$\int_{\mathbb{C} \setminus D(w, \delta_n)} B_n^{\langle w \rangle}(z) dA(z) \lesssim n\delta_n^3 = o(1).$$

Proof. We write $D_n = D(w, \delta_n)$ and notice that

$$\int_{\mathbb{C} \setminus D_n} B_n^{\langle w \rangle} = 1 - \int_{D_n} B_n^{\langle w \rangle} = 1 - \int_{D_n} H_n^{\langle w \rangle} + \int_{D_n} (H_n^{\langle w \rangle} - B_n^{\langle w \rangle}) = \int_{\mathbb{C} \setminus D_n} H_n^{\langle w \rangle} + \int_{D_n} (H_n^{\langle w \rangle} - B_n^{\langle w \rangle}).$$

The statement now follows from Lemma 3.3. \square

Proposition 3.5. *If v is uniformly Lipschitz continuous on \mathbb{C} , then $\epsilon_n^1[v] \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. We represent the error term as follows:

$$\epsilon_n^1[v] = \int_{(w)} u_n(w) F_n(w) \quad ; \quad F_n(w) = \int_{(z)} \left[\frac{v(z) - v(w)}{z - w} - \partial v(w) \right] B_n^{\langle w \rangle}(z).$$

By the assumption that v is globally Lipschitzian, we have that

$$\left| \int_{\{\delta(w) < 2\delta_n\}} u_n(w) F_n(w) \right| \lesssim \int_{\{\delta(w) < 2\delta_n\}} u_n(w) = o(1).$$

If $\delta(w) > 2\delta_n$, then

$$|F_n(w)| \lesssim \left| \int_{z \in D(w, \delta_n)} \left[\frac{v(z) - v(w)}{z - w} - \partial v(w) \right] B_n^{\langle w \rangle}(z) \right| + \text{const.} \int_{z \notin D(w, \delta_n)} B_n^{\langle w \rangle}(z),$$

where the last term is $o(1)$ by Corollary 3.4. Meanwhile, the integral over $D(w, \delta_n)$ is bounded by

$$\left| \int_{z \in D(w, \delta_n)} \left[\frac{v(z) - v(w)}{z - w} - \partial v(w) \right] H_n^{\langle w \rangle}(z) \right| + \text{const.} \int_{D(w, \delta_n)} |B_n^{\langle w \rangle}(z) - H_n^{\langle w \rangle}(z)|,$$

where we can neglect the second term (see Lemma 3.3). Finally,

$$\frac{v(z) - v(w)}{z - w} - \partial v(w) = \bar{\partial} v(w) \frac{\bar{z} - \bar{w}}{z - w} + o(1),$$

(this is where we use the assumption $v \in C^1(S)$), so the bound of the first term is $o(1)$ by the radial symmetry of the heat kernel. \square

3.3. Second error term. We shall prove the following proposition.

Proposition 3.6. *If v is uniformly Lipschitzian, then*

$$\epsilon_n^2[v] := -\frac{1}{2} \int \bar{\partial} v \frac{D_n^2}{n} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

The proof will involve certain estimates of the function

$$D_n(z) = \int_{\mathbb{C}} \frac{\mathbf{K}_n(\zeta) - \mathbf{K}_n^\#(\zeta)}{z - \zeta} d^2 \zeta.$$

(Here $\mathbf{K}^\#(\zeta) = nu \cdot 1_S$.) It will be convenient to split the integral into two parts:

$$D_n(z) = C_n(z) + R_n(z) := \left(\int_{B_n} + \int_{\mathbb{C} \setminus B_n} \right) \frac{\mathbf{K}_n(\zeta) - \mathbf{K}_n^\#(\zeta)}{z - \zeta} d^2 \zeta,$$

where

$$B_n = \{z : \delta(z) < 2\delta_n\}.$$

By Theorem 3.2 and Lemma 3.1 we have $|\mathbf{K}_n - \mathbf{K}_n^\#| \lesssim 1$ in $\mathbb{C} \setminus B_n$, and therefore

$$|R_n| \lesssim 1.$$

Hence we only need to estimate C_n , the Cauchy transform of a real measure supported in B_n – a narrow "ring" around ∂S . We start with a simple uniform bound.

Lemma 3.7. *The following estimate holds,*

$$\|D_n\|_{L^\infty} \lesssim \sqrt{n} \log^3 n.$$

Proof. This follows from the trivial bound $|\mathbf{K}_n - \mathbf{K}_n^\#| \lesssim n$ and the following estimate of the integral

$$\int_{B_n} \frac{d^2 \zeta}{|z - \zeta|}.$$

Without loosing generality, we can assume that $z = 0$ and replace B_n by the rectangle $|x| < 1$, $|y| < \delta_n$. We have

$$\int_{B_n} \frac{d^2 \zeta}{|\zeta|} = \int_{-1}^1 dx \int_{-\delta_n}^{\delta_n} \frac{dy}{\sqrt{x^2 + y^2}} = I + II,$$

where I is the integral over $\{|x| < \delta_n\}$ and II is the remaining term. Passing to polar coordinates we get

$$I \asymp \int_0^{\delta_n} \frac{r dr}{r} = \delta_n,$$

and

$$II \lesssim \delta_n \int_{\delta_n}^1 \frac{dx}{x} \asymp \delta_n |\log \delta_n|.$$

□

Lemma 3.7 gives us the following estimate of the second error term,

$$(3.2) \quad |\epsilon_n^2[v]| \lesssim \log^6 n \|v\|_{\text{Lip}},$$

which comes rather close but is still weaker than what we want. Our strategy will be to use (3.2) and iterate the argument with Ward's identity. This will give a better estimate in the *interior* of the droplet.

Lemma 3.8. *We have that*

$$|C_n(z)| \lesssim \frac{\log^6 n}{\delta(z)^3}, \quad z \in S.$$

Proof. Let ψ be a function of Lipschitz norm less than 1 supported inside the droplet. Then we have

$$\varepsilon_n^1[\psi] \lesssim 1, \quad \varepsilon_n^2[\psi] \lesssim \log^6 n,$$

where the constants don't depend on ψ . (The first estimate follows from Proposition 3.5, and the second one is just (3.2)). This means that the error $\varepsilon_n := \varepsilon_n^1 + \varepsilon_n^2$ in the identity (2.12) is bounded by $\log^6 n$ for all such ψ , i.e. (since $\partial Q = \partial \check{Q}$ a.e. on S),

$$\left| \int \psi D_n \Delta Q \right| = O(1) + |\varepsilon_n[\psi]| \lesssim \log^6 n,$$

and therefore,

$$\left| \int \psi C_n \Delta Q \right| \lesssim \log^6 n$$

For $z \in S_n$, we now set $2\delta = \delta(z)$ and consider the function

$$\psi(\zeta) = \max \left\{ \frac{\delta - |\zeta - z|}{\Delta Q(\zeta)}, 0 \right\}.$$

Then ψ has Lipschitz norm $\asymp 1$, and by analyticity of C_n we have the mean value identity

$$\int \psi C_n \Delta Q = 2\pi C_n(z) \int_0^\delta (\delta - r) r dr = \pi C_n(z) \delta^3 / 3.$$

We conclude that $|C_n(z)| \lesssim \delta^{-3} \log^6 n$. □

Finally, we need an estimate of C_n in the *exterior* of the droplet. This will be done in the next subsection by reflecting the previous interior estimate in the curve $\Gamma := \partial S$.

Let us fix some sufficiently small positive number, e.g. $\varepsilon = \frac{1}{10}$ will do, and define

$$\gamma_n = n^{-\varepsilon}.$$

Denote

$$\Gamma_n = \{\zeta + \gamma_n \nu(\zeta) : \zeta \in \Gamma\},$$

where $\nu(\zeta)$ is the unit normal vector to Γ at $\zeta \in \Gamma$ pointing outside from S . We will write $\text{int}\Gamma_n$ and $\text{ext}\Gamma_n$ for the respective components of $\mathbb{C} \setminus \Gamma_n$. In the following, the notation $a < b$ will mean inequality up to a multiplicative constant factor times some power of $\log n$ (thus e.g. $1 < \log^2 n$).

Let $L^2(\Gamma_n)$ be the usual L^2 space of functions on Γ_n with respect to arclength measure. We will use the following lemma.

Lemma 3.9. *Suppose that v is uniformly Lipschitzian. Then*

$$\|C_n\|_{L^2(\Gamma_n)}^2 < n\gamma_n.$$

Given this estimate, we can complete the proof of Proposition 3.6 as follows.

Proof of Proposition 3.6. Applying Green's formula to the expression for $\varepsilon_n^2[v]$ (see (2.11)), we find that

$$|\varepsilon_n^2[v]| \lesssim \frac{1}{n} \|D_n\|_{L^2(\text{int}\Gamma_n)}^2 + \frac{1}{n} \|D_n\|_{L^2(\Gamma_n)}^2,$$

where $\text{int}\Gamma_n$ is the γ_n -neighbourhood of the droplet. The second term is taken care of by Lemma 3.9. To estimate the first term denote

$$A_n = \{\delta(z) < \gamma_n\}.$$

The area of A_n is $\asymp \gamma_n$, and in $S \setminus A_n$ we have $|D_n(z)| < \gamma_n^{-3}$ (Lemma 3.8). We now apply the uniform bound $|D_n| < \sqrt{n}$ in A_n (Lemma 3.7). It follows that

$$\|D_n\|_{L^2(\text{int}\Gamma_n)}^2 = \int_{A_n} + \int_{S \setminus A_n} < n|A_n| + \gamma_n^{-6},$$

whence

$$\|D_n\|_{L^2(\text{int}\Gamma_n)}^2 = o(n).$$

This finishes the proof of the proposition. \square

3.4. Proof of Lemma 3.9. Let us first establish the following fact:

$$(3.3) \quad |\text{Im} [\nu(\zeta) C_n(\zeta + \gamma_n \nu(\zeta))]| < \sqrt{n\gamma_n}, \quad \zeta \in \Gamma.$$

Proof. Without loss of generality, assume that $\zeta = 0$ and $\nu(\zeta) = i$.

The tangent to Γ at 0 is horizontal, so Γ is the graph of $y = y(x)$ where $y(x) = O(x^2)$ as $x \rightarrow 0$. We will show that

$$(3.4) \quad |\text{Re}[C_n(i\gamma_n) - C_n(-i\gamma_n)]| < \sqrt{n\gamma_n}.$$

This implies the desired estimate (3.3), because by Lemma 3.8

$$|C_n(-i\gamma_n)| < \gamma_n^{-3} \leq \sqrt{n\gamma_n}.$$

To prove (3.4) we notice that

$$I := \text{Re}[C_n(i\gamma_n) - C_n(-i\gamma_n)] = \int_{B_n} \text{Re} \left[\frac{1}{z - i\gamma_n} - \frac{1}{z + i\gamma_n} \right] \rho_n(z) dA(z),$$

where we have put $\rho_n = \mathbf{K}_n - \mathbf{K}_n^\#$, viz. $|\rho_n| < n$.

We next subdivide the belt $B_n = \{\delta(z) < 2\delta_n\}$ into two parts:

$$B'_n = B_n \cap \{|x| \leq \sqrt{\gamma_n}\}, \quad B''_n = B_n \setminus B'_n.$$

Clearly,

$$(3.5) \quad |I| \lesssim n \int_{B'_n} \left| \frac{1}{z - i\gamma_n} - \frac{1}{\bar{z} - i\gamma_n} \right| + n \int_{B''_n} \left| \frac{1}{z - i\gamma_n} - \frac{1}{z + i\gamma_n} \right|.$$

The integral over B'_n in the right hand side of (3.5) is estimated by

$$\int_{B'_n} \frac{|y|}{x^2 + \gamma_n^2} \lesssim |B'_n| \asymp \delta_n \sqrt{\gamma_n}$$

because if $z = x + iy \in B'_n$ then $|y| \lesssim x^2 + \delta_n \leq x^2 + \gamma_n^2$.

We estimate the integral over B_n'' in (3.5) by

$$\int_{B_n''} \asymp \int_{B_n''} \frac{\gamma_n}{x^2} < \delta_n \gamma_n \int_{\sqrt{\gamma_n}}^1 \frac{dx}{x^2} \asymp \delta_n \sqrt{\gamma_n}.$$

It follows that

$$|I| < n \delta_n \sqrt{\gamma_n} < \sqrt{n \gamma_n}.$$

This establishes (3.4), and, as a consequence, (3.3). \square

To finish the proof of Lemma 3.9 we denote by $\nu_n(\cdot)$ the outer unit normal of Γ_n . Using (3.3) and Lemma 3.7 we deduce that

$$|\operatorname{Im} \nu_n C_n| < \sqrt{n \gamma_n} \quad \text{on } \Gamma_n.$$

Next let \mathbb{D}_* be the exterior of the closed unit disk and consider the conformal map

$$\phi_n : \operatorname{ext}(\Gamma_n) \rightarrow \mathbb{D}_*, \quad \infty \mapsto \infty.$$

We put

$$F_n = \frac{\phi_n C_n}{\phi_n'}.$$

Then F_n is analytic in $\operatorname{ext}(\Gamma_n)$ including infinity, and we have

$$(3.6) \quad \|\operatorname{Im} F_n\|_{L^2(\Gamma_n)}^2 < \sqrt{n \gamma_n}.$$

To see this note that

$$\operatorname{Im} F_n = \frac{\operatorname{Im}[\nu_n C_n]}{|\phi_n'|},$$

and recall that we have assumed that Γ is regular (A3), which means that $|\phi_n'|$ is bounded below by a positive constant.

Now note that $\phi_n(z)/\phi_n'(z) = r_n z + O(1)$ as $z \rightarrow \infty$, where the r_n are uniformly bounded. This gives

$$F_n(\infty) = r_n \int_{B_n} (\mathbf{K}_n - \mathbf{K}_n^\#) = r_n \int_{\mathbb{C} \setminus B_n} (\mathbf{K}_n^\# - \mathbf{K}_n) = O(1),$$

where we have used Lemma 3.1 and Theorem 3.2 to bound the integrand. Therefore, by (3.6), since the harmonic conjugation operator is bounded on the Hardy space H^2 ,

$$\|\operatorname{Re} F_n\|_{L^2(\Gamma_n)}^2 \asymp \|\operatorname{Im} F_n\|_{L^2(\Gamma_n)}^2 + O(1) < \sqrt{n \gamma_n}.$$

This completes the proof of Lemma 3.9.

4. PROOF OF THE MAIN FORMULA

In this section we will use the limit form of Ward's identity (Corollary 2.5) to derive our main formula (1.8): for every test function f the limit $\tilde{v}(f) := \lim_{n \rightarrow \infty} \tilde{v}_n(f)$ exists and equals

$$(4.1) \quad \tilde{v}(f) = \frac{1}{8\pi} \left[\int_S \Delta f + \int_S f \Delta L + \int_{\partial S} f \mathcal{N}(L^S) ds \right] + \frac{1}{2\pi} \int_{\mathbb{C}} \nabla f^S \cdot \nabla h^S.$$

4.1. Decomposition of f . The following statement uses our assumption that ∂S is a (real analytic) Jordan curve.

Lemma 4.1. *Let $f \in C_0^\infty(\mathbb{C})$. Then f has the following representation:*

$$f = f_+ + f_- + f_0,$$

where

- (i) all three functions are smooth on \mathbb{C} ;
- (ii) $\bar{\partial}f_+ = 0$ and $\partial f_- = 0$ in $\mathbb{C} \setminus S$;
- (iii) $f_\pm = O(1)$ at ∞ ;
- (iv) $f_0 = 0$ on ∂S .

Proof. Consider the inverse conformal maps

$$\phi : \mathbb{D}_* \rightarrow \mathbb{C} \setminus S, \quad \psi : \mathbb{C} \setminus S \rightarrow \mathbb{D}_*, \quad \infty \mapsto \infty,$$

where $\mathbb{D}_* = \{|z| > 1\}$. On the unit circle \mathbb{T} , we have

$$F := f \circ \phi = \sum_{-\infty}^{\infty} a_n \zeta^n \in C^\infty(\mathbb{T}).$$

The functions

$$F_+(z) = \sum_{-\infty}^0 a_n z^n, \quad F_-(z) = \sum_1^{\infty} \frac{a_n}{\bar{z}^n}, \quad (z \in \mathbb{D}_*)$$

are C^∞ up to the boundary so we can extend them to some smooth functions F_\pm in \mathbb{C} . The conformal map ψ also extends to a smooth function $\psi : \mathbb{C} \rightarrow \mathbb{C}$. It follows that

$$f_\pm := F_\pm \circ \psi \in C_0^\infty(\mathbb{C}),$$

and f_\pm satisfy (ii)–(iii). Finally, we set

$$f_0 = f - f_+ - f_-.$$

□

Conclusion. It is enough to prove the main formula (4.1) only for functions of the form $f = f_+ + f_- + f_0$ as in the last lemma with an *additional* assumption that f_0 is supported inside any given neighborhood of the droplet S .

Indeed, either side of the formula (4.1) will not change if we "kill" f_0 outside the neighborhood. The justification is immediate by Lemma 3.1.

In what follows we will choose a neighborhood O of S such that the potential Q is real analytic, strictly subharmonic in O , and

$$\partial Q \neq \partial \check{Q} \quad \text{in } O \setminus S,$$

and will assume $\text{supp}(f_0) \subset O$.

4.2. The choice of the vector field in Ward's identity. We will now compute the limit

$$\tilde{v}(f) := \lim \tilde{v}_n(f)$$

(and prove its existence) in the case where

$$f = f_+ + f_0.$$

To apply the limit Ward identity

$$(4.2) \quad \frac{2}{\pi} \int_{\mathbb{C}} [v \partial \bar{\partial} Q + \bar{\partial} v (\partial Q - \partial \check{Q})] \check{D}_n \rightarrow -\frac{1}{2} \sigma(\partial v) - 2\sigma(v \partial h), \quad (n \rightarrow \infty),$$

(see Corollary 2.5), we set

$$v = v_+ + v_0,$$

where

$$v_0 = \frac{\bar{\partial} f_0}{\partial \bar{\partial} Q} \cdot 1_S + \frac{f_0}{\partial Q - \partial \check{Q}} \cdot 1_{\mathbb{C} \setminus S},$$

and

$$v_+ = \frac{\bar{\partial} f_+}{\partial \bar{\partial} Q} \cdot 1_S.$$

This gives

$$v = \frac{\bar{\partial} f}{\partial \bar{\partial} Q} \cdot 1_S + \frac{f_0}{\partial Q - \partial \check{Q}} \cdot 1_{\mathbb{C} \setminus S}.$$

But in $\mathbb{C} \setminus \partial S$ we have

$$v \partial \bar{\partial} Q + \bar{\partial} v \cdot \partial(Q - \check{Q}) = \bar{\partial} f,$$

so comparing with (2.9), we find that

$$(4.3) \quad -2\tilde{v}_n(f) = \frac{2}{\pi} \int_{\mathbb{C}} [v \partial \bar{\partial} Q + \bar{\partial} v (\partial Q - \partial \check{Q})] \check{D}_n.$$

However, to justify that (4.2) holds, we must check that v satisfies the conditions (i)-(iii) of Corollary 2.5.

Lemma 4.2. *The vector field v defined above is $\text{Lip}(\mathbb{C})$ and the restrictions of v to S and to $S_* := (\mathbb{C} \setminus S) \cup \partial S$ are C^∞ .*

Proof. We need to check the following items:

(i) $v|_S$ is smooth, and (i') $v|_S$ is smooth;

(ii) v_0 is continuous on ∂S , and (ii') same for v_+ .

The items (i') and (ii') are of course trivial. (E.g., $\bar{\partial} f_+ \cdot 1_S = \bar{\partial} f_+$.)

Proof of (i). We have $v = f_0/g$ in $\mathbb{C} \setminus S$ where $g = \partial Q - \partial \check{Q}$. Since the statement is local, we consider a conformal map ϕ that takes a neighbourhood of a boundary point in S onto a neighbourhood of a point in \mathbb{R} and takes (parts of) ∂S to \mathbb{R} . If we denote $F = f_0 \circ \phi$ and $G = g \circ \phi$, then $F = 0$ and $G = 0$ on \mathbb{R} . Moreover, G is real analytic with non-vanishing derivative G_y . Thus it is enough to check that

$$H(x, y) = \frac{F(x, y)}{y}$$

has bounded derivatives of all orders. We will go through the details for H, H_y, H_{yy}, \dots . Applying the same argument to H_x , we get the boundedness of the derivatives $H_x, H_{xy}, H_{yy}, \dots$, etc.

Let us show, e.g., that $H' := H_y$ is bounded. We have

$$H' = \frac{yF' - F}{y^2} = \frac{y(F'_0 + O(y)) - (yF'_0 + O(y^2))}{y^2} = O(1),$$

where $F'_0 := F'(\cdot, 0)$ and all big O 's are uniform in x . (They come from the bounds for the derivatives of F .) Similarly,

$$H'' = \frac{y^2 F'' - 2yF' + 2F}{y^3}.$$

The numerator is

$$y^2(F''_0 + O(y)) - 2y(F'_0 + yF''_0 + O(y^2)) + 2(yF'_0 + \frac{1}{2}y^2F''_0 + O(y^3)) = O(y^3),$$

etc. (We can actually stop here because we only need C^2 smoothness to apply Theorem 2.3.)

Proof of (ii). Let $n = n(\zeta)$ be the exterior unit normal with respect to S . We have

$$f_0(\zeta + \delta n) \sim \delta \partial_n f_0(\zeta) = 2\delta \cdot (\bar{\partial} f_0)(\zeta) \cdot \overline{n(\zeta)}, \quad \text{as } \delta \downarrow 0.$$

Similarly, if $g := \partial Q - \partial \check{Q}$, so $g = 0$ on ∂S and $\bar{\partial} g = \partial \bar{\partial} Q$ in $\mathbb{C} \setminus S$, then

$$g(\zeta + \delta n) \sim \delta \partial_n g(\zeta) = 2\delta \cdot (\bar{\partial} g)(\zeta) \cdot \overline{n(\zeta)}, \quad \text{as } \delta \downarrow 0.$$

It follows that

$$\frac{f_0(\zeta + \delta n)}{g(\zeta + \delta n)} \sim \frac{\bar{\partial} f_0(\zeta)}{\partial \bar{\partial} Q(\zeta)}, \quad (\delta \downarrow 0),$$

which proves the continuity of v_0 . □

We have established that $v = v_0 + v_+$ satisfies conditions (i)-(iii) of Corollary 2.5. Thus the convergence in (4.2) holds, and by (4.3) we conclude the following result.

Corollary 4.3. *If $f = f_0 + f_+$, then*

$$\tilde{v}(f) = \frac{1}{4} \sigma(\partial v) + \sigma(v \partial h).$$

4.3. Conclusion of the proof.

(a). Let us now consider the general case

$$f = f_+ + f_0 + f_-.$$

By the last corollary we have

$$\tilde{v}(f_+) = \frac{1}{4} \sigma(\partial v_+) + \sigma(v_+ \partial h), \quad v_+ := \frac{\bar{\partial} f_+}{\partial \bar{\partial} Q} \cdot 1_S.$$

Using complex conjugation we get a similar expression for $\tilde{v}(f_-)$:

$$\tilde{v}(f_-) = \frac{1}{4} \sigma(\bar{\partial} v_-) + \sigma(v_- \bar{\partial} h), \quad v_- := \frac{\partial f_-}{\partial \bar{\partial} Q} \cdot 1_S.$$

Indeed,

$$\tilde{v}(f_-) = \overline{\tilde{v}(f_-)} = \overline{\frac{1}{4}\sigma(\partial\bar{v}_-) + \sigma(\bar{v}_-\partial h)} = \frac{1}{4}\sigma(\bar{\partial}v_-) + \sigma(v_-\bar{\partial}h).$$

(Recall that h is real-valued.)

Summing up we get

$$(4.4) \quad \nu(f) = \frac{1}{4} \left[\sigma(\partial v_0) + \sigma(\partial v_+) + \sigma(\bar{\partial} v_-) \right]$$

and

$$(4.5) \quad \tilde{v}(f) - \nu(f) = \sigma(v_+\partial h) + \sigma(v_0\partial h) + \sigma(v_-\bar{\partial}h).$$

(b) *Computation of $\nu(f)$.* Recall that

$$d\sigma(z) = \frac{1}{2\pi} \Delta Q(z) \mathbf{1}_S(z) d^2z \quad , \quad L = \log \Delta Q.$$

Using (4.4) we compute

$$\begin{aligned} \nu(f) &= \frac{1}{2\pi} \int_S \partial \left(\frac{\bar{\partial} f_0 + \bar{\partial} f_+}{\partial \bar{\partial} Q} \right) \partial \bar{\partial} Q + \frac{1}{2\pi} \int_S \bar{\partial} \left(\frac{\partial f_-}{\partial \bar{\partial} Q} \right) \partial \bar{\partial} Q \\ &= \frac{1}{2\pi} \int_S \partial \left(\frac{\bar{\partial} f_0}{\partial \bar{\partial} Q} \cdot \partial \bar{\partial} Q \right) - \frac{1}{2\pi} \int_S \frac{\bar{\partial} f_0}{\partial \bar{\partial} Q} \partial(\partial \bar{\partial} Q) + \frac{1}{2\pi} \int_S \partial \left(\frac{\bar{\partial} f_+}{\partial \bar{\partial} Q} \right) \partial \bar{\partial} Q + \frac{1}{2\pi} \int_S \bar{\partial} \left(\frac{\partial f_-}{\partial \bar{\partial} Q} \right) \partial \bar{\partial} Q \\ &= \frac{1}{2\pi} \int_S \partial \bar{\partial} f - \frac{1}{2\pi} \int_S \bar{\partial} f_0 \partial L - \frac{1}{2\pi} \int_S \bar{\partial} f_+ \partial L - \frac{1}{2\pi} \int_S \partial f_- \bar{\partial} L \end{aligned}$$

At this point, let us modify L outside some neighborhood of S to get a smooth function with compact support. We will still use the notation L for the modified function. The last expression clearly does not change as a result of this modification. We can now transform the integrals involving L as follows:

$$\begin{aligned} - \int_S \bar{\partial} f_0 \partial L - \int_{\mathbb{C}} \bar{\partial} f_+ \partial L - \int_{\mathbb{C}} \partial f_- \bar{\partial} L &= \int_S f_0 \partial \bar{\partial} L + \int_{\mathbb{C}} (f_+ + f_-) \partial \bar{\partial} L \\ &= \int_S f \partial \bar{\partial} L + \int_{\mathbb{C} \setminus S} f^S \partial \bar{\partial} L, \end{aligned}$$

and we conclude that

$$\nu(f) = \frac{1}{8\pi} \left[\int_S \Delta f + \int_S f \Delta L + \int_{\mathbb{C} \setminus S} f^S \Delta L \right]$$

Note. The formula for $\nu(f)$ was stated in this form in [3].

Let us finally express the last integral in terms of Neumann's jump. We have

$$\begin{aligned}
 \int_{\mathbb{C} \setminus S} f^S \Delta L &= \int_{\mathbb{C} \setminus S} (f^S \Delta L - L \Delta f^S) \\
 &= \int_{\partial S} (f^S \cdot \partial_{n^*} L - \partial_{n^*} f^S \cdot L^S) ds \\
 &= \int_{\partial S} (f^S \cdot \partial_{n^*} L - f^S \cdot \partial_{n^*} L^S) ds \\
 &= \int_{\partial S} f \mathcal{N}(L^S) ds
 \end{aligned}$$

In conclusion,

$$(4.6) \quad \nu(f) = \frac{1}{8\pi} \left[\int_S \Delta f + \int_S f \Delta L + \int_{\partial S} f \mathcal{N}(L^S) \right].$$

(c) *Computation of $[\tilde{\nu}(f) - \nu(f)]$.* Using the identity (4.5), we can deduce that

$$\begin{aligned}
 \tilde{\nu}(f) - \nu(f) &= \frac{2}{\pi} \left[\int_S \bar{\partial} f_+ \partial h + \int_S \partial f_- \bar{\partial} h + \int_S \bar{\partial} f_0 \partial h \right] \\
 &= \frac{1}{2\pi} \int \nabla f^S \cdot \nabla h^S.
 \end{aligned}$$

This is because

$$\int_S \bar{\partial} f_+ \partial h = \int_{\mathbb{C}} \bar{\partial} f_+ \partial h = -\frac{1}{4} \int_{\mathbb{C}} f_+ \Delta h = \frac{1}{4} \int_{\mathbb{C}} \nabla f_+ \cdot \nabla h,$$

and similarly

$$\int_S \partial f_- \bar{\partial} h = \frac{1}{4} \int_{\mathbb{C}} \nabla f_- \cdot \nabla h.$$

On the other hand,

$$\int_S \bar{\partial} f_0 \partial h = -\frac{1}{4} \int_S f_0 \Delta h = \frac{1}{4} \int_S \nabla f_0 \cdot \nabla h.$$

Therefore,

$$\tilde{\nu}(f) - \nu(f) = \frac{1}{2\pi} \left[\int_S \nabla f \cdot \nabla h + \int_{\mathbb{C} \setminus S} \nabla f^S \cdot \nabla h \right],$$

and this is equal to

$$\frac{1}{2\pi} \int_{\mathbb{C}} \nabla f^S \cdot \nabla h = \frac{1}{2\pi} \int_{\mathbb{C}} \nabla f^S \cdot \nabla h^S.$$

Applying (4.6) we find that

$$\tilde{\nu}(f) = \frac{1}{8\pi} \left[\int_S \Delta f + \int_S f \Delta L + \int_{\partial S} f \mathcal{N}(L^S) \right] + \frac{1}{2\pi} \int_{\mathbb{C}} \nabla f^S \cdot \nabla h^S,$$

and the main formula (4.1) has been completely established. q.e.d.

APPENDIX: BULK ASYMPTOTICS FOR THE CORRELATION KERNEL

Polynomial Bergman spaces. For a suitable (extended) real valued function ϕ , we denote by L_ϕ^2 the space normed by $\|f\|_\phi^2 = \int_{\mathbb{C}} |f|^2 e^{-2\phi}$. We denote by A_ϕ^2 the subspace of L_ϕ^2 consisting of a.e. entire functions; $\mathcal{P}_n(e^{-2\phi})$ denotes the subspace consisting of analytic polynomials of degree at most $n - 1$.

Now consider a potential Q , real analytic and strictly subharmonic in some neighborhood of the droplet S , and subject to the usual growth condition. We put

$$\tilde{Q} \equiv \tilde{Q}_n = Q - \frac{1}{n}h,$$

where h is a smooth bounded real function.

We denote by K the reproducing kernel for the space $\mathcal{P}_n(e^{-2nQ})$, and write $K_w(z) = K(z, w)$. The corresponding orthogonal projection is denoted by

$$P_n : L_{nQ}^2 \rightarrow \mathcal{P}_n(e^{-2nQ}) \quad : \quad f \mapsto (f, K_w)_{nQ}.$$

The map $\tilde{P}_n : L_{n\tilde{Q}}^2 \rightarrow \mathcal{P}_n(e^{-2n\tilde{Q}})$ is defined similarly, using the reproducing kernel \tilde{K} for the space $\mathcal{P}_n(e^{-2n\tilde{Q}})$.

We define approximate kernels and Bergman projection as follows. In the case $h = 0$, the well-known first order approximation inside the droplet is given by the expression

$$K_w^\#(z) = \frac{2}{\pi} (\partial_1 \partial_2 Q)(z, \bar{w}) n e^{2nQ(z, \bar{w})},$$

where $Q(\cdot, \cdot)$ is the complex analytic function of two variables satisfying

$$Q(w, \bar{w}) = Q(w).$$

If the perturbation $h \neq 0$ is a real-analytic function, we can just replace Q by \tilde{Q} in this expression. Note that in this case, the analytic extension $h(\cdot, \cdot)$ satisfies

$$h(z, \bar{w}) = h(w) + (z - w)\partial h(w) + \dots, \quad (z \rightarrow w).$$

This motivates the definition of the approximate Bergman kernel in the case where h is only a smooth function: we set

$$\tilde{K}_w^\#(z) = K_w^\#(z) e^{-2h_w(z)},$$

where

$$h_w(z) := h(w) + (z - w)\partial h(w).$$

The approximate Bergman projection is defined accordingly:

$$\tilde{P}_n^\# f(w) = (f, \tilde{K}_w^\#)_{n\tilde{Q}}.$$

The kernels $\tilde{K}_n^\#(z, w)$ do not have the Hermitian property. The important fact is that they are analytic in z .

Proof of Theorem 3.2. We shall prove the following estimate.

Lemma 4.4. *If $z \in S$, $\delta(z) > 2\delta_n$, and if $|z - w| < \delta_n$, then*

$$|\tilde{K}_w(z) - \tilde{K}_w^\#(z)| \lesssim e^{nQ(z)} e^{nQ(w)}.$$

Before we prove the lemma, we use it to conclude the proof of Theorem 3.2. Recall that

$$\tilde{\mathbf{K}}_n(z, w) = \tilde{K}_w(z) e^{-n\tilde{Q}(z)} e^{-n\tilde{Q}(w)}.$$

If we define

$$\tilde{\mathbf{K}}_n^\#(z, w) = \tilde{K}_w^\#(z) e^{-n\tilde{Q}(z)} e^{-n\tilde{Q}(w)},$$

then by Lemma 4.4,

$$\tilde{\mathbf{K}}_n(z, w) = \tilde{\mathbf{K}}_n^\#(z, w) + O(1).$$

On the other hand, we have

$$\tilde{\mathbf{K}}_n^\#(z, w) = \mathbf{K}_n^\#(z, w) e^{h(z)+h(w)-2h_w(z)},$$

so

$$|\tilde{\mathbf{K}}_n^\#(z, w)| = |\mathbf{K}_n^\#(z, w)| (1 + O(|w - z|^2)) = |\mathbf{K}_n^\#(z, w)| + O(1).$$

It follows that

$$|\tilde{\mathbf{K}}_n(z, w)| = |\mathbf{K}_n^\#(z, w)| + O(1),$$

as claimed in Theorem 3.2. \square

It remains to prove Lemma 4.4.

Lemma 4.5. *If f is analytic and bounded in $D(z; 2\delta_n)$ then*

$$|f(z) - \tilde{P}_n^\#(\chi_z f)(z)| \lesssim \frac{1}{\sqrt{n}} e^{nQ(z)} \|f\|_{nQ}.$$

Here $\chi = \chi_z$ is a cut-off function with $\chi = 1$ in $D(z; 3\delta_n/2)$ and $\chi = 0$ outside $D(z; 2\delta_n)$ satisfying $\|\bar{\partial}\chi\|_2 \asymp 1$.

Proof. Wlog, $z = 0$, so $\tilde{P}_n^\#(\chi f)(0)$ is the integral

$$I^\# = \frac{1}{\pi} \int \chi(\zeta) \cdot f(\zeta) \cdot 2(\partial_1 \partial_2 Q)(0, \bar{\zeta}) \cdot e^{2[h(\zeta)-h(0)-\bar{\zeta}\bar{\partial}h(0)]} \cdot ne^{-2n[Q(\zeta, \bar{\zeta})-Q(0, \bar{\zeta})]}.$$

Since

$$\bar{\partial}_{\bar{\zeta}} \left[e^{-2n[Q(\zeta, \bar{\zeta})-Q(0, \bar{\zeta})]} \right] = -2[\partial_2 Q(\zeta, \bar{\zeta}) - \partial_2 Q(0, \bar{\zeta})] ne^{-2n[Q(\zeta, \bar{\zeta})-Q(0, \bar{\zeta})]},$$

we can rewrite the expression as follows:

$$I^\# = -\frac{1}{\pi} \int \frac{1}{\bar{\zeta}} f(\zeta) \chi(\zeta) A(\zeta) B(\zeta) \bar{\partial} \left[e^{-2n[Q(\zeta, \bar{\zeta})-Q(0, \bar{\zeta})]} \right],$$

where

$$A(\zeta) = \frac{\zeta (\partial_1 \partial_2 Q)(0, \bar{\zeta})}{\partial_2 Q(\zeta, \bar{\zeta}) - \partial_2 Q(0, \bar{\zeta})},$$

and

$$B(\zeta) = e^{2[h(\zeta)-h(0)-\bar{\zeta}\bar{\partial}h(0)]}.$$

A trivial but important observation is that

$$A, B = O(1) \quad , \quad \bar{\partial}A = O(|\zeta|) \quad , \quad \bar{\partial}B = O(|\zeta|),$$

where the O -constants have uniform bounds throughout.

Integrating by parts we get

$$I^\# = f(0) + \epsilon_1 + \epsilon_2,$$

where

$$\epsilon_1 = \int \frac{f(\bar{\partial}\chi) AB}{\bar{\zeta}} e^{-2n[Q(\zeta, \bar{\zeta}) - Q(0, \bar{\zeta})]} \quad , \quad \epsilon_2 = \int \frac{f\chi \bar{\partial}(AB)}{\bar{\zeta}} e^{-2n[Q(\zeta) - Q(0, \bar{\zeta})]}.$$

Using that

$$|\epsilon_1| \lesssim \frac{1}{\delta_n} \int |f| |\bar{\partial}\chi| e^{-2n[Q(\zeta) - \operatorname{Re} Q(0, \bar{\zeta})]} \quad , \quad |\epsilon_2| \lesssim \int \chi |f| e^{-2n[Q(\zeta) - \operatorname{Re} Q(0, \bar{\zeta})]},$$

and noting that Taylor's formula gives

$$e^{-n[Q(\zeta) - 2 \operatorname{Re} Q(0, \bar{\zeta})]} \lesssim e^{nQ(0) - cn|\zeta|^2}, \quad (c \sim \Delta Q(0) > 0),$$

we find, by the Cauchy–Schwarz inequality, (since $|\zeta| \geq \delta_n$ when $\bar{\partial}\chi(\zeta) \neq 0$)

$$|\epsilon_1| e^{-nQ(0)} \lesssim \frac{e^{-cn\delta_n^2}}{\delta_n} \|f\|_{nQ} \|\bar{\partial}\chi\|_{L^2} \lesssim \frac{1}{\sqrt{n}} \|f\|_{nQ}$$

and

$$|\epsilon_2| e^{-nQ(0)} \lesssim \|f\|_{nQ} \left(\int e^{-nc|\zeta|^2} \right)^{1/2} \lesssim \frac{1}{\sqrt{n}} \|f\|_{nQ}.$$

The proof is finished. □

Suppose now that $\operatorname{dist}(z, \mathbb{C} \setminus S) \geq 2\delta_n$ and $|w - z| \leq \delta_n$.

From Lemma 4.5, we conclude that

$$(4.7) \quad \left| \tilde{K}_w(z) - \tilde{P}_n \left[\chi_z \tilde{K}_w^\# \right] (z) \right| \lesssim e^{nQ(z)} e^{nQ(w)}.$$

This is because

$$\tilde{P}_n^\# \left[\chi_z \tilde{K}_z \right] (w) = (\chi_z \tilde{K}_z, \tilde{K}_w^\#)_{n\bar{Q}} = \overline{(\chi_z \tilde{K}_w^\#, \tilde{K}_z)_{n\bar{Q}}} = \overline{\tilde{P}_n \left[\chi \tilde{K}_w^\# \right] (z)},$$

so

$$\left| \tilde{K}_w(z) - \tilde{P}_n \left[\chi_z \tilde{K}_w^\# \right] (z) \right| = \left| \tilde{K}_z(w) - \tilde{P}_n^\# \left[\chi_z \tilde{K}_z \right] (w) \right|$$

and because (cf. [2], Section 3)

$$\|\tilde{K}_z\| = \sqrt{\tilde{K}_z(z)} \lesssim \sqrt{n} e^{nQ(z)}.$$

On the other hand, we will prove that

$$(4.8) \quad \left| \tilde{K}_z^\#(w) - \tilde{P}_n \left[\chi_z \tilde{K}_z^\# \right] (w) \right| \lesssim e^{nQ(z)} e^{nQ(w)},$$

which combined with (4.7) proves Lemma 4.4. The verification of the last inequality is the same as in [5] or [1], depending on the observation that $L_{nQ}^2 = L_{n\bar{Q}}^2$ with equivalence of norms. We give a detailed argument, for completeness.

For given smooth f , consider u , the L_{nQ}^2 -minimal solution to the problem

$$(4.9) \quad \bar{\partial}u = \bar{\partial}f \quad \text{and} \quad u - f \in \mathcal{P}_{n-1}.$$

Since $\|u\|_{n\bar{Q}} \lesssim \|u\|_{nQ}$, the $L_{n\bar{Q}}^2$ -minimal solution \tilde{u} to the problem (4.9) satisfies $\|\tilde{u}\|_{n\bar{Q}} \leq C\|u\|_{nQ}$. We next observe that $P_n f$ is related to the L_{nQ}^2 -minimal solution u to the problem (4.9) by $u = f - P_n f$.

We write

$$u(\zeta) = \chi_z(\zeta) \tilde{K}_w^\#(\zeta) - P_n [\chi_z \tilde{K}_w^\#](\zeta),$$

i.e., u is the L^2_{nQ} -minimal solution to (4.9) for $f = \chi_z \cdot \tilde{K}_w^\#$. Let us verify that

$$(4.10) \quad \|u\|_{nQ} \lesssim \left\| \bar{\partial} (\chi_z \cdot \tilde{K}_w^\#) \right\|_{nQ}.$$

To prove this, we put

$$2\phi(\zeta) = 2\check{Q}(\zeta) + n^{-1} \log(1 + |\zeta|^2),$$

and consider the function v_0 , the $L^2_{n\phi}$ -minimal solution to the problem $\bar{\partial}v = \bar{\partial}(\chi_z \cdot \tilde{K}_w^\#)$. Notice that ϕ is strictly subharmonic on \mathbb{C} . By Hörmander's estimate (e.g. [12], p. 250)

$$\|v_0\|_{n\phi}^2 \lesssim \int_{\mathbb{C}} \left| \bar{\partial} (\chi_z \cdot \tilde{K}_w^\#) \right|^2 \frac{e^{-2n\phi}}{n\Delta\phi}.$$

Since χ_z is supported in S , we hence have

$$\|v_0\|_{n\phi} \lesssim \frac{1}{\sqrt{n}} \left\| \bar{\partial} (\chi_z \cdot \tilde{K}_w^\#) \right\|_{nQ}.$$

We next observe that by the growth assumption on Q near infinity, we have an estimate $n\phi \leq nQ + \text{const.}$ on \mathbb{C} , which gives $\|v_0\|_{nQ} \lesssim \|v_0\|_{n\phi}$. It yields that

$$\|v_0\|_{nQ} \lesssim \frac{1}{\sqrt{n}} \left\| \bar{\partial} (\chi_z \cdot \tilde{K}_w^\#) \right\|_{nQ}.$$

But $v_0 - \chi_z \cdot \tilde{K}_w^\#$ belongs to the weighted Bergman space $A^2_{n\phi}$. Since $2n\phi(\zeta) = (n+1) \log|\zeta|^2 + O(1)$ as $\zeta \rightarrow \infty$, the latter space coincides with \mathcal{P}_{n-1} as sets. This shows that v_0 solves the problem (4.9). Since $\|u\|_{nQ} \leq \|v_0\|_{nQ}$, we then obtain (4.10).

By norms equivalence, (4.10) implies that

$$(4.11) \quad \|\tilde{u}\|_{n\check{Q}} \lesssim \frac{1}{\sqrt{n}} \left\| \bar{\partial} (\chi_z \cdot \tilde{K}_w^\#) \right\|_{n\check{Q}},$$

where

$$\tilde{u} = \chi_z \tilde{K}_w^\# - \tilde{P}_n [\chi_z \tilde{K}_w^\#]$$

is the $L^2_{n\check{Q}}$ -minimal solution to (4.9) with $f = \chi_z \tilde{K}_w^\#$.

We now set out to prove the pointwise estimate

$$(4.12) \quad |\tilde{u}(z)| \lesssim n e^{-cn\delta_n^2} e^{n(Q(z)+Q(w))}.$$

To prove this, we first observe that

$$\bar{\partial}\tilde{u}(\zeta) = \bar{\partial}(\chi_z \cdot \tilde{K}_w^\#)(\zeta) = \bar{\partial}\chi_z(\zeta) \cdot \tilde{K}_w^\#(\zeta),$$

whence, by the form of $\tilde{K}_w^\#$ and Taylor's formula,

$$\left| \bar{\partial}\tilde{u}(\zeta) \right|^2 e^{-2nQ(\zeta)} \lesssim n^2 \left| \bar{\partial}\chi_z(\zeta) \right|^2 e^{2n(Q(w)-c|\zeta-w|^2)}$$

with a positive constant $c \sim \Delta Q(z)$. Since $|\zeta - w| \geq \delta_n/2$ when $\bar{\partial}\chi(\zeta) \neq 0$, it yields

$$\left| \bar{\partial} (\chi_z \cdot \tilde{K}_w^\#) \right|^2 e^{-2nQ(\zeta)} \lesssim n^2 \left| \bar{\partial}\chi_z(\zeta) \right|^2 e^{2nQ(w)-cn\delta_n^2}.$$

We have shown that

$$\left\| \bar{\partial} \left(\chi_z \cdot \tilde{K}_w^\# \right) \right\|_{nQ} \lesssim n e^{-nc\delta_n^2} e^{nQ(w)}.$$

In view of the estimate (4.11), we then have

$$\|\tilde{u}\|_{nQ} \lesssim \sqrt{n} e^{-nc\delta_n^2} e^{nQ(w)}.$$

Since \tilde{u} is analytic in $D(z; 1/\sqrt{n})$ we can now invoke the simple estimate (e.g. [2], Lemma 3.2)

$$|\tilde{u}(z)|^2 e^{-2nQ(z)} \lesssim n \|\tilde{u}\|_{nQ}^2$$

to get

$$|\tilde{u}(z)| \lesssim n e^{-nc\delta_n^2} e^{n(Q(z)+Q(w))}.$$

This gives (4.8), and finishes the proof of Lemma 4.4. \square

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